

Angular momentum and Horn's problem

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Abstract

We prove a conjecture made in [C1]: given an n -body central configuration X_0 in the euclidean space E of dimension $2p$, let $Im\mathcal{F}$ be the set of ordered real p -tuples $\{\nu_1, \nu_2, \dots, \nu_p\}$ such that $\{\pm i\nu_1, \pm i\nu_2, \dots, \pm i\nu_p\}$ is the spectrum of the angular momentum of some (periodic) relative equilibrium motion of X_0 in E . Then $Im\mathcal{F}$ is a convex polytope. The proof consists in showing that there exist two, generically $(p-1)$ -dimensional, convex polytopes \mathcal{P}_1 and \mathcal{P}_2 in \mathbb{R}^p such that $\mathcal{P}_1 \subset Im\mathcal{F} \subset \mathcal{P}_2$ and that these two polytopes coincide.

\mathcal{P}_1 , introduced in [C1], is the set of spectra corresponding to the hermitian structures J on E which are “adapted” to the symmetries of the inertia matrix S_0 ; it is associated with Horn’s problem for the sum of $p \times p$ real symmetric matrices with spectra σ_- and σ_+ whose union is the spectrum of S_0 .

\mathcal{P}_2 is the orthogonal projection onto the set of “hermitian spectra” of the polytope \mathcal{P} associated with Horn’s problem for the sum of $2p \times 2p$ real symmetric matrices having each the same spectrum as S_0 .

The equality $\mathcal{P}_1 = \mathcal{P}_2$ follows directly from a deep combinatorial lemma, proved in [FFLP], which characterizes those of the sums $C = A+B$ of two $2p \times 2p$ real symmetric matrices A and B with the same spectrum, which are hermitian for some hermitian structure.

1 Origin of the problem: N -body relative equilibria and their angular momenta

We recall here the results of [AC, C1, C2] which are needed in order to understand the mechanical origin of the purely algebraic conjecture solved in the present paper: given a configuration $x_0 = (\vec{r}_1, \dots, \vec{r}_N) \in E^N$ of N punctual positive masses in the euclidean space E , a *rigid motion* of the configuration under Newton’s attraction is a motion in which the mutual distances $\|\vec{r}_i - \vec{r}_j\|$ between the bodies stay constant. It is proved

in [AC] (see also [C2]) that such a motion is necessarily a *relative equilibrium*. This implies that the motion takes place in a space of even dimension $2p$, which can be supposed to coincide with E , and that, in a galilean frame fixing the center of mass at the origin, it is of the form $x(t) = (e^{\Omega t} \vec{r}_1, e^{\Omega t} \vec{r}_2, \dots, e^{\Omega t} \vec{r}_N)$, where Ω is a $2p \times 2p$ -antisymmetric endomorphism of the euclidean space E which is non degenerate. Choosing an orthonormal basis where Ω is normalized, this amounts to saying that there exists a hermitian structure on the space E of motion and an orthogonal decomposition $E \equiv \mathbb{C}^p = \mathbb{C}^{k_1} \times \dots \times \mathbb{C}^{k_r}$ such that

$$x(t) = (x_1(t), \dots, x_r(t)) = (e^{i\omega_1 t} x_1, \dots, e^{i\omega_r t} x_r),$$

where x_m is the orthogonal projection on \mathbb{C}^{k_m} of the N -body configuration x and the action of $e^{i\omega_m t}$ on x_m is the diagonal action on each body of the projected configuration. Such quasi-periodic motions exist only for very special configurations, called *balanced configurations* (see [AC, C2] for their characterization). The most degenerate balanced configurations are the *central configurations* for which all the frequencies ω_i are the same; this means that $\Omega = \omega J$, with J a hermitian structure on E , and the motion is

$$x(t) = (\vec{r}_1(t), \dots, \vec{r}_N(t)) = e^{i\omega t} x_0 = (e^{i\omega t} \vec{r}_1, \dots, e^{i\omega t} \vec{r}_N)$$

in the hermitian space $E \equiv \mathbb{C}^p$; in particular, it is periodic. In a space of dimension at most 3, E is necessarily of dimension 2 and the configuration of any relative equilibrium is central.

Given a configuration $x = (\vec{r}_1, \dots, \vec{r}_N)$ and a configuration of velocities $y = \dot{x} = (\vec{v}_1, \dots, \vec{v}_N)$, both with center of mass at the origin: $\sum_{k=1}^N m_k \vec{r}_k = \sum_{k=1}^N m_k \vec{v}_k = 0$, the *angular momentum* of (x, y) is the bivector $C = \sum_{k=1}^N m_k \vec{r}_k \wedge \vec{v}_k$. If we represent x and y by the $2p \times N$ matrices X and Y whose i th column are respectively made of the components (r_{1i}, \dots, r_{2pi}) and (v_{1i}, \dots, v_{2pi}) of \vec{r}_i and \vec{v}_i in an orthonormal basis of E and if $M = \text{diag}(m_1, \dots, m_N)$, this bivector is represented by the antisymmetric matrix (*we use the french convention tX for the transposed of X*)

$$C = -XM^tY + YM^tX \quad \text{with coefficients} \quad c_{ij} = \sum_{k=1}^N m_k (-r_{ik}v_{jk} + r_{jk}v_{ik}).$$

The dynamics of a solid body is determined by its *inertia tensor* (with respect to its center of mass), represented in the case of a point masses configuration X by the symmetric matrix

$$S = XM^tX \quad \text{with coefficients} \quad s_{ij} = \sum_{k=1}^N m_k r_{ik}r_{jk},$$

whose trace is the *moment of inertia of the configuration x with respect to its center of mass*. In particular, the angular momentum of a relative equilibrium solution $X(t) = e^{t\Omega} X_0$ is represented by the antisymmetric matrix $C = S_0\Omega + \Omega S_0$, where $S_0 = X_0 M^t X_0$. Restricting to the case of central configurations, that is $\Omega = \omega J$, and making $\omega = 1$, we consider

in what follows the spectrum of J -skew-hermitian matrices of the form $S_0J + JS_0$ or, what amounts to the same, the spectrum of J -hermitian matrices¹ of the form $J^{-1}S_0J + S_0$.

In the following, we identify E with \mathbb{R}^{2p} by the choice of some orthonormal basis. \mathbb{R}^{2p} is endowed with its canonical basis $e_i = (0, \dots, 1, \dots, 0)$ and its canonical euclidean scalar product $x \cdot y = \sum_{i=1}^{2p} x_i y_i$; this allows identifying linear endomorphisms of $E = \mathbb{R}^{2p}$ and $2p \times 2p$ matrices with real coefficients. When we say that J is a hermitian structure, we mean that the standard euclidean structure is given and that J is a complex structure which is orthogonal.

2 The frequency map

We recall the definition, given in [C1], of the *frequency map* \mathcal{F} from the set of hermitian structures on \mathbb{R}^{2p} to the positive Weyl chamber $W_p^+ \subset \mathbb{R}^p$: given some $2p \times 2p$ real symmetric matrix S_0 , we consider the mapping $J \mapsto J^{-1}S_0J + S_0$ from the space of hermitian structures on E to the set of $2p \times 2p$ real symmetric matrices. We are only interested in the spectra of these matrices, hence choosing an orientation for J is harmless and we shall consider only those of the form $J = R^{-1}J_0R$, where J_0 is the “standard” structure defined by $J_0 = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$ and $R \in SO(2p)$.

Two elements R' and R'' of $SO(2p)$ defining the same J if and only if there exists an element $U \in U(p)$ such that $R'' = UR'$, it follows that the space of (oriented) hermitian structures is identified to the homogeneous space $U(p) \backslash SO(2p)$. The symmetric matrix $J^{-1}S_0J + S_0$ is J -hermitian, that is, it commutes with J . This implies that its spectrum is real, of the form $\{\nu_1, \nu_2, \dots, \nu_p\}$ if considered as a $p \times p$ complex matrix (for the identification of \mathbb{R}^{2p} to \mathbb{C}^p defined by J) and of the form $\{(\nu_1, \nu_2, \dots, \nu_p), (\nu_1, \nu_2, \dots, \nu_p)\}$ if considered as a $2p \times 2p$ real matrix (see the next section for the trivial proof).

Definition 1 *The frequency mapping*

$$\mathcal{F} : U(p) \backslash SO(2p) \rightarrow W_p^+ = \{(\nu_1, \dots, \nu_p) \in \mathbb{R}^p, \nu_1 \geq \dots \geq \nu_p\}$$

associates to each hermitian structure J the ordered spectrum (ν_1, \dots, ν_p) of the J -hermitian matrix $J^{-1}S_0J + S_0$.

3 Hermitian spectra

Lemma 1 *Let $C : \mathbb{R}^{2p} \rightarrow \mathbb{R}^{2p}$ be a symmetric endomorphism. The following assertions are equivalent:*

- 1) *There exists a hermitian structure $J = R^{-1}J_0R$ such that C be J -hermitian (i.e. $JC = CJ$);*
- 2) *The spectrum $\sigma(C)$ of C is of the form*

$$\sigma(C) = \{(\nu_1, \nu_2, \dots, \nu_p), (\nu_1, \nu_2, \dots, \nu_p)\}.$$

¹Notice that this is the same as the spectrum of the J_0 -hermitian matrix $\Sigma = J_0^{-1}SJ_0 + S$, where $S = RS_0R^{-1}$, which was considered in [C1].

Proof. Let $J = R^{-1}J_0R$; the mapping C is J -hermitian if and only if RCR^{-1} is J_0 -hermitian. This is equivalent to the existence of an isomorphism $U \in U(p) \subset SO(2p)$ such that

$$U^{-1}RCR^{-1}U = \text{diag}((\nu_1, \dots, \nu_p), (\nu_1, \dots, \nu_p)).$$

Conversely, the identity

$$R^{-1}CR = \text{diag}((\nu_1, \dots, \nu_p), (\nu_1, \dots, \nu_p))$$

implies the commutation of $R^{-1}CR$ with J_0 and hence the commutation of C with $J = R^{-1}J_0R$.

Notations. We shall call *hermitian* the spectra of this form and *the diagonal* the linear subspace Δ of W_{2p}^+ defined by

$$\Delta = \{(\mu_1 \geq \mu_2 \geq \dots \geq \mu_{2p}), \mu_1 = \mu_2, \mu_3 = \mu_4, \dots, \mu_{2p-1} = \mu_{2p}\}.$$

Hence the ordered hermitian spectra are the ones belonging to Δ .

4 Two convex polytopes

Let $S_0 : \mathbb{R}^{2p} \rightarrow \mathbb{R}^{2p}$ be a symmetric endomorphism with spectrum

$$\sigma(S_0) = \{\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{2p}\}.$$

To S_0 we associate the subsets \mathcal{P}_1 and \mathcal{P}_2 of \mathbb{R}^p (in fact of the positive Weyl chamber W_p^+ of ordered real p -tuples), defined as follows:

1) \mathcal{P}_1 is the set of ordered spectra

$$\sigma(c) = \{\nu_1 \geq \nu_2 \geq \dots \geq \nu_p\}$$

of symmetric endomorphisms c of \mathbb{R}^p of the form $c = a + b$, where $a : \mathbb{R}^p \rightarrow \mathbb{R}^p$ and $b : \mathbb{R}^p \rightarrow \mathbb{R}^p$ are arbitrary symmetric endomorphisms with respective spectra

$$\sigma(a) = \sigma_- := \{\sigma_1, \sigma_3, \dots, \sigma_{2p-1}\}, \quad \sigma(b) = \sigma_+ := \{\sigma_2, \sigma_4, \dots, \sigma_{2p}\};$$

2) \mathcal{P}_2 is the set of p -tuples $\{\nu_1 \geq \nu_2 \geq \dots \geq \nu_p\}$ such that

$$\{(\nu_1, \nu_2, \dots, \nu_p), (\nu_1, \nu_2, \dots, \nu_p)\}$$

is the spectrum of some symmetric endomorphism C of \mathbb{R}^{2p} of the form $C = A + B$, where $A : \mathbb{R}^{2p} \rightarrow \mathbb{R}^{2p}$ and $B : \mathbb{R}^{2p} \rightarrow \mathbb{R}^{2p}$ are arbitrary symmetric endomorphisms with the same spectrum

$$\sigma(A) = \sigma(B) = \sigma(S_0).$$

In other words, identifying canonically the diagonal Δ with \mathbb{R}^p , one can write

$$\mathcal{P}_2 = \mathcal{P} \cap \Delta,$$

where \mathcal{P} is the $(2p - 1)$ -dimensional Horn polytope which describes the ordered spectra of sums $C = A + B$ of $2p \times 2p$ real symmetric matrices A, B with the same spectrum as S_0 .

Lemma 2 \mathcal{P}_1 and \mathcal{P}_2 are both contained in the hyperplane of \mathbb{R}^p with equation

$$\sum_{i=1}^p \nu_i = \sum_{j=1}^{2p} \sigma_j.$$

Moreover, \mathcal{P}_1 and \mathcal{P}_2 are both $(p-1)$ -dimensional convex polytopes and

$$\mathcal{P}_1 \subset \text{Im}\mathcal{F} \subset \mathcal{P}_2.$$

Proof. The first identity comes from the additivity of the trace function. The fact that both \mathcal{P}_1 , \mathcal{P} and hence $\mathcal{P}_2 = \mathcal{P} \cap \Delta$, are convex polytopes is a general fact coming from the interpretation of the Horn problem as a moment map problem. Finally, the second inclusion comes from the very definition of \mathcal{F} and the first comes from Lemma 1 and the following identity, where σ_- and σ_+ are considered as $p \times p$ diagonal matrices and $\rho \in SO(p)$:

$$\left\{ \begin{array}{l} \begin{pmatrix} \sigma_- & 0 \\ 0 & \sigma_+ \end{pmatrix} + \begin{pmatrix} 0 & -\rho^{-1} \\ \rho & 0 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_- & 0 \\ 0 & \sigma_+ \end{pmatrix} \begin{pmatrix} 0 & -\rho^{-1} \\ \rho & 0 \end{pmatrix} \\ = \begin{pmatrix} \sigma_- + \rho^{-1}\sigma_+\rho & 0 \\ 0 & \rho\sigma_-\rho^{-1} + \sigma_+ \end{pmatrix}. \end{array} \right.$$

Remark. The choice of the partition $\sigma = \sigma_- \cup \sigma_+$ of σ is dictated by the following theorem, which proves that any other partition of σ into two subsets with p elements will lead to a smaller polytope \mathcal{P}_1 :

Theorem 3 ([FFLP] Proposition 2.2) *Let A and B be $p \times p$ Hermitian matrices. Let $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{2p}$ be the eigenvalues of A and B arranged in descending order. Then there exist Hermitian matrices \tilde{A} and \tilde{B} with eigenvalues $\sigma_1 \geq \sigma_3 \geq \dots \geq \sigma_{2p-1}$ and $\sigma_2 \geq \sigma_4 \geq \dots \geq \sigma_{2p}$ respectively, such that $\tilde{A} + \tilde{B} = A + B$.*

This was used in [C1] to prove that $\mathcal{P}_1 = \text{Im}\mathcal{A}$ is the image under the frequency map \mathcal{F} of the *adapted* hermitian structures, i.e. those J which send some p -dimensional subspace of \mathbb{R}^{2p} generated by eigenvectors of S_0 onto its orthogonal.

5 The projection property

In this section, we prove the

Theorem 4 *The two polytopes \mathcal{P}_1 and \mathcal{P}_2 coincide.*

Corollary 5 $\text{Im}\mathcal{F} = \mathcal{P}_1 = \text{Im}\mathcal{A}$. *In other words, the image by the frequency map \mathcal{F} of the adapted structures is already the full image $\text{Im}\mathcal{F}$.*

We need recall the inductive definition of the Horn inequalities which define the Horn polytopes (see [F]). For a sum $a + b = c$ of symmetric $p \times p$ matrices with respective (ordered in decreasing order) spectra

$$\alpha = (\alpha_1, \dots, \alpha_p), \quad \beta = (\beta_1, \dots, \beta_p), \quad \gamma = (\gamma_1, \dots, \gamma_p),$$

they read

$$(* IJK) \quad \forall r < p, \forall (I, J, K) \in T_r^p, \quad \sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j,$$

where T_r^p (notation of [F], noted LR_r^p by reference to Littlewood-Richardson coefficients in [FFLP]) is defined as follows: let U_r^p be the set of triples (I, J, K) of subsets of cardinal r of $\{1, 2, \dots, p\}$ such that

$$\sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + \frac{r(r+1)}{2}.$$

Then set $T_1^p = U_1^p$ and define recursively T_r^p by

$$T_r^p = \left[(I, J, K) \in U_r^p, \forall s < r, \forall (F, G, H) \in T_s^r, \right. \\ \left. \sum_{f \in F} i_f + \sum_{g \in G} j_g \leq \sum_{h \in H} k_h + \frac{s(s+1)}{2} \right]$$

An immediate computation gives the following

Lemma 6 *Let*

$$\begin{cases} I_2 &= (2i_1 - 1, 2i_2 - 1, \dots, 2i_r - 1, 2j_1, 2j_2, \dots, 2j_r), \\ J_2 &= (2i_1 - 1, 2i_2 - 1, \dots, 2i_r - 1, 2j_1, 2j_2, \dots, 2j_r), \\ K_2 &= (2k_1 - 1, 2k_1, 2k_2 - 1, 2k_2, \dots, 2k_r - 1, 2k_r), \end{cases}$$

Then $(I_2, J_2, K_2) \in U_{2r}^{2p}$.

Proof. It suffices to check that

$$2 \left[\sum_{i \in I} (2i - 1) + \sum_{j \in J} (2j) \right] = \sum_{k \in K} (2k - 1) + \sum_{k \in K} 2k + \frac{2r(2r+1)}{2}.$$

Now, comes the key fact:

Theorem 7 ([FFLP], lemma 1.18) *For any triple (I, J, K) in T_r^p , the triple (I_2, J_2, K_2) is in T_{2r}^{2p}*

The proof of this theorem, which concerns the so-called “domino-decomposable Young diagrams”, is based on a version of the Littlewood-Richardson rule due to Carré and Leclerc [CL].

It implies that, for any a sum $A + B = C$ of real symmetric $2p \times 2p$ matrices with respective (ordered in decreasing order) spectra

$$\hat{\alpha} = (\hat{\alpha}_1, \dots, \hat{\alpha}_{2p}), \quad \hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_{2p}), \quad \hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_{2p}),$$

$(* I_2, J_2, K_2)$ holds, that is

$$\sum_{k \in K} (\hat{\gamma}_{2k-1} + \hat{\gamma}_{2k}) \leq \sum_{i \in I} (\hat{\alpha}_{2i-1} + \hat{\beta}_{2i-1}) + \sum_{j \in J} (\hat{\alpha}_{2j} + \hat{\beta}_{2j}).$$

In particular, if

$$\hat{\alpha} = \hat{\beta} = \sigma = (\sigma_1, \sigma_2, \dots, \sigma_{2p}),$$

we get that

$$\sum_{k \in K} \frac{\hat{\gamma}_{2k-1} + \hat{\gamma}_{2k}}{2} \leq \sum_{i \in I} \sigma_{2i-1} + \sum_{j \in J} \sigma_{2j}.$$

Note that the mapping

$$(\hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_{2p-1}, \hat{\gamma}_{2p}) \mapsto \left(\frac{\hat{\gamma}_1 + \hat{\gamma}_2}{2}, \frac{\hat{\gamma}_1 + \hat{\gamma}_2}{2}, \dots, \frac{\hat{\gamma}_{2p-1} + \hat{\gamma}_{2p}}{2}, \frac{\hat{\gamma}_{2p-1} + \hat{\gamma}_{2p}}{2} \right)$$

is the orthogonal projection of the ordered set $(\hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_{2p-1}, \hat{\gamma}_{2p})$ on the *diagonal* Δ of \mathbb{R}^{2p} defined by the equations $\hat{\gamma}_1 = \hat{\gamma}_2, \dots, \hat{\gamma}_{2p-1} = \hat{\gamma}_{2p}$, that is on the subset of “hermitian” spectra. Hence a paraphrase of the above theorem is

Theorem 8 *Let $C = A + B$ be the sum of two $2p \times 2p$ real symmetric matrices with the same spectrum $\{\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{2p}\}$. If $\{\nu_1 \geq \dots \geq \nu_p\}$ is the orthogonal projection on the diagonal $\Delta \equiv \mathbb{R}^p$ of the spectrum $\{\hat{\gamma}_1 \geq \hat{\gamma}_2 \geq \dots \geq \hat{\gamma}_{2p}\}$ of C , that is if $\nu_k = \frac{\hat{\gamma}_{2k-1} + \hat{\gamma}_{2k}}{2}$, the triple of ordered spectra*

$$\alpha = (\sigma_1, \sigma_3, \dots, \sigma_{2p-1}), \beta = (\sigma_2, \sigma_4, \dots, \sigma_{2p}), \gamma = (\nu_1, \nu_2, \dots, \nu_p)$$

*satisfies the Horn inequality $(*I, J, K)$.*

This implies the following extremal property of the subset of “hermitian” spectra:

Corollary 9 *The orthogonal projection on the diagonal Δ of the $(2p-1)$ -dimensional Horn polytope $\mathcal{P} \subset \mathbb{R}^{2p}$ associated with the spectra*

$$\sigma(A) = \sigma(B) = \{\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{2p}\}$$

coincides with the $(p-1)$ -dimensional Horn polytope $\mathcal{P}_1 \in \mathbb{R}^p$ associated with the spectra

$$\sigma(a) = (\sigma_1, \sigma_3, \dots, \sigma_{2p-1}), \sigma(b) = (\sigma_2, \sigma_4, \dots, \sigma_{2p}).$$

In particular, the intersection $\mathcal{P}_2 = \mathcal{P} \cap \Delta$ corresponding to the hermitian spectra, that is those such that $\hat{\gamma}_1 = \hat{\gamma}_2, \dots, \hat{\gamma}_{2p-1} = \hat{\gamma}_{2p}$, coincides with \mathcal{P}_1 . Indeed, \mathcal{P}_2 coincides with the projection of \mathcal{P} , which itself coincides with \mathcal{P}_1 .

Remark. The equality $\text{Im}\mathcal{F} = \mathcal{P}_2$ implies the following

Corollary 10 *Let $C : \mathbb{R}^{2p} \rightarrow \mathbb{R}^{2p}$ be the sum $C = A + B$ of two symmetric endomorphisms A and B with the same spectrum $\sigma(A) = \sigma(B) = \sigma(S_0)$. Then C is J -hermitian for some hermitian structure J on \mathbb{R}^{2p} if and only if it is conjugate by an element of $SO(2p)$ to a matrix of the form $S_0 + \tilde{J}^{-1}S_0\tilde{J}$, where \tilde{J} is a hermitian structure on \mathbb{R}^{2p} .*

Note. The proof of the above results has been written by the first author after he was convinced by the numerical experiments made by the second author that the equality $\mathcal{P}_1 = \mathcal{P}_2$ was plausible when $p = 3$ and more precisely that not only the intersection $\mathcal{P}_2 = \mathcal{P} \cap \Delta$ but also the orthogonal projection of the Horn polytope \mathcal{P} on Δ was contained in \mathcal{P}_1 after the canonical identification of Δ with \mathbb{R}^p . This led first to a proof when $p = 2$ or 3, obtained by coping directly with Horn’s inequalities and then to the discovery that the general case followed from a lemma which turned out to be exactly the lemma 1.18 of [FFLP]. The numerical experiments are described in the next section.

6 Numerical experiments

The numerical checking of the conjecture that $Im\mathcal{F} = \mathcal{P}_1$, was made on the matrix $S_0 = \frac{1}{32}\text{diag}\{13, 8, 5, 3, 2, 1\}$, whose spectrum satisfies the inequalities in [C1] (section 8). We wrote a program in TRIP [GL11] producing different rotation matrices $R \in SO(2p)$ in a random way. Starting from the canonical basis $\xi = \{\xi_1, \dots, \xi_m\}$, $m = p(2p - 1)$, of $\mathfrak{so}(2p)$, we created a list containing the m one-parameter subgroups $G_i(t) = e^{t\xi_i} \subset SO(2p)$. We created a second list of m random values $[t_i]_{i=1}^m$ and a random permutation $[1, 2, \dots, m] \rightarrow [i_1, i_2, \dots, i_m]$. The random rotation matrix was defined as

$$R = \prod_{j=1}^m G_{i_j}(t_j), \quad 0 \leq t_i \leq 2\pi. \quad (1)$$

The program which plots \mathcal{P}_1 is very simple (the fact that we replaced the conjugation of J_0 by the conjugation of S_0 is immaterial and comes from the formulation of the conjecture in [C1]):

```
create  $S_0$  and  $J_0$ 
for  $i = 1$  to  $N_{max}$  do
  create  $R$  and  $R^{-1}$ ;
   $S = RS_0R^{-1}$ ;
   $C = S - J_0SJ_0$ ;
   $lst = \text{eigenvalues}(C)$ ;
  plot (  $lst[5]$ ,  $lst[3]$ ,  $lst[1]$  );
end for.
```

We have assigned the value $N_{max} = 25000$ obtaining the results shown in Figure 1. The figure shows also the simplex $\gamma_1 + \gamma_2 + \gamma_3 = 1$ and its intersection with W_3^+ .

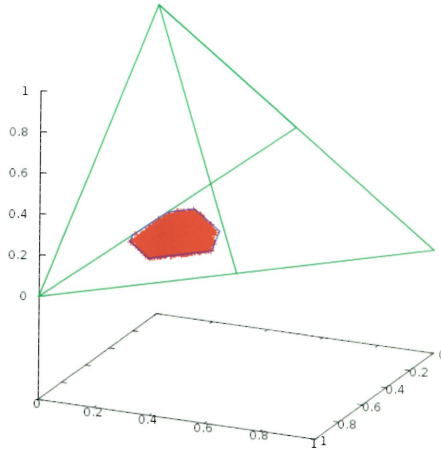


Figure 1: $\mathcal{P}_1 = Im\mathcal{A}$: 25000 random adapted hermitian structures

The modified algorithm to estimate the shape of $\mathcal{P}_2 = \mathcal{P} \cap \Delta$ in W_3^+ is similar. For a random R in $SO(6)$, the ordered spectrum $\text{spec}(C) = (\gamma_1, \dots, \gamma_6)$ of $C = S_0 + R^{-1}S_0R$ is projected orthogonally onto the diagonal Δ by the map $\pi_\Delta : \mathbb{R}^6 \rightarrow \Delta$:

$$(\gamma_1, \gamma_2, \dots, \gamma_6) \mapsto \left(\frac{\gamma_1 + \gamma_2}{2}, \frac{\gamma_3 + \gamma_4}{2}, \frac{\gamma_5 + \gamma_6}{2} \right).$$

At first, when $\text{spec}(C)$ was ε -close to Δ i.e., if $\sum_{k=1}^3 |\gamma_{2k-1} - \gamma_{2k}|^2 < 2\varepsilon^2$ for ε small, the projected point was plotted in green; otherwise it was plotted in red. No particular pattern was found for the green points meanwhile the red ones were all contained in \mathcal{P}_1 . The algorithm to plot $\pi_\Delta(\mathcal{P})$ is

```

create  $S_0$ 
for  $i = 1$  to  $N_{max}$  do
  create  $R$  and  $R^{-1}$ ;
   $C = S_0 + R^{-1}S_0R$ ;
   $lst = \text{eigenvalues}(C)$ ;
  sort(  $lst$  );
  plot  $\left( \frac{lst[6]+lst[5]}{2}, \frac{lst[4]+lst[3]}{2}, \frac{lst[2]+lst[1]}{2} \right)$ ;
end for.

```

The results of the projection $\pi_\Delta(\text{spec}(C))$ for 50000 random rotation matrices are shown in Figure 2.

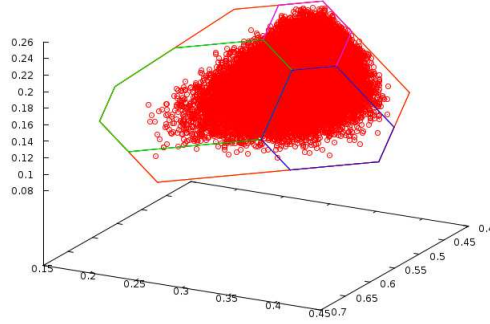


Figure 2: Projection of \mathcal{P} : 50000 random rotations

The matrix S_0 and hence the polytope \mathcal{P}_1 are the same as in Figure 1 (the interior lines correspond to the polytopes associated to different partitions of the spectrum of S_0 , as depicted in the corresponding figure in [C1]). Recall that the polytope \mathcal{P} has dimension 5; this explains that in order to get a better filling one should have taken many more points. This was not done because the evidence was sufficiently convincing to ask for a proof.

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